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THE ZEROES OF NONNEGATIVE HOLOMORPHIC CURVATURE OPERATORS

BY

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ABSTRACT. Here, we study the structure of points in a holomorphic Grassmann's submanifold where the holomorphic sectional curvature assumes its minimum and maximum. For spaces of nonnegative holomorphic sectional curvature we study the set of points on which it assumes the value zero. We show that the minimum and maximum sets of holomorphic sectional curvature are the intersections of a holomorphic Grassmann's submanifold with linear complex holomorphic subspaces of type (1, 1).

Thorpe, [4], completely describes the structure of the sets of points in the Grassmann manifold of tangent 2-planes at a point where the riemannian sectional curvature assumes its minimum and maximum. In particular, for spaces of nonnegative curvature he describes the set of points in the Grassmann manifold where the riemannian sectional curvature assumes the value zero.

The holomorphic sectional curvatures are invariants of the Hermitian structure weaker than the riemannian sectional curvature. The study of these invariants is very interesting as can be seen by the abundant bibliography on this subject.

If M is an almost Kaehler manifold with almost complex structure J and $m \in M$, then both the set of holomorphic 2-planes at m (planes invariant under J) and the set of antiholomorphic 2-planes at m (planes P such that $v \in P$ implies $Jv \perp P$) are intersections with the Grassmann manifold of linear subspaces of $\Lambda^2(V)$ where $V = T_mM$ is the tangent space of M at m. Indeed, the automorphism J of V induces a curvature operator, also denoted by J, on V by $J(u \wedge v) = Ju \wedge Jv$ $(u, v \in V)$ and one checks that the sectional curvature of J assumes its maximum value 1 on holomorphic 2-planes and its minimum value 0 on antiholomorphic 2-planes [4].

In this article, proceeding in the same way as Thorpe, [4], for the real case, we describe the structure of the sets of points in a holomorphic Grassmann's submanifold

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$$G = \{\alpha \in \Lambda^{1,1}(V^{\mathbf{C}})/g(\alpha, \bar{\alpha}) = 1, \alpha + \bar{\alpha} = 0, \alpha \text{ decomposable}\},$$

where the holomorphic sectional curvature assumes its minimum and maximum. Similarly to the real case, for spaces of nonnegative holomorphic curvature we describe the set of points where the holomorphic sectional curvature assumes the value zero and we show that the minimum and maximum sets of holomorphic sectional curvature are intersections with G of linear complex subspaces of $\Lambda^{1,1}$. We have omitted proofs of several steps where the obvious generalization to complex cases of Thorpe's proof is valid.

1. **Decomposition of the space of complex curvature operators.** Let V be (the tangent space of a Kaehler manifold) a $2n$-dimensional real vector space with a complex structure J and a Hermitian inner product g. Then g can be extended uniquely to a complex symmetric bilinear form, denoted also by g, of $V^{\mathbf{C}} = V \otimes_{\mathbf{R}} \mathbf{C} = V^{1,0} \oplus V^{0,1}$ and it satisfies the following conditions:

(1) $g(\bar{u}, \bar{v}) = \overline{g(u, v)}$ for $u, v \in V^{\mathbf{C}}$,

(2) $g(u, \bar{u}) > 0$ for all nonzero $u \in V^{\mathbf{C}}$,

(3) $g(u, \bar{v}) = 0$ for all $u \in V^{1,0}$ and $v \in V^{0,1}$.

We denote by $\Lambda^{p,q} V^{\mathbf{C}}$ the subspace of $\Lambda V^{\mathbf{C}}$ spanned by $\alpha \wedge \beta$, where $\alpha \in \Lambda^p V^{1,0}$ and $\beta \in \Lambda^q V^{0,1}$. For r an integer $\geqslant 0$, let $\Lambda^r(V^{\mathbf{C}}) = \Sigma_{p+q=r} \Lambda^{p,q} V^{\mathbf{C}}$ denote the space of complex r-vectors of $V^{\mathbf{C}}$ equipped with a complex symmetric bilinear form given by

$$g(u_1 \wedge \cdots \wedge u_r, v_1 \wedge \cdots \wedge v_r) = \det\{g(u_i, v_j)\}, \quad u_i, v_j \in V^{\mathbf{C}}.$$

Let G denote the Grassmann's submanifold of holomorphic planes

$$G = \{\alpha \in \Lambda^{1,1}/\alpha = u \wedge v, u \in V^{1,0}, v \in V^{0,1}, g(\alpha, \bar{\alpha}) = 1, \alpha + \bar{\alpha} = 0\}.$$

Although the Grassmann submanifold of holomorphic (invariant under J) planes is usually regarded as a collection of subspaces of V, it will be useful for our purposes to define it as a collection G of subspaces of $V^{\mathbf{C}}$ which are in bijective correspondence with the elements of the usual Grassmann holomorphic submanifold.

Let R denote the space of selfadjoint linear operators on $\Lambda^{1,1}$, equipped with a complex symmetric bilinear form:

$$g(R, S) = \operatorname{Tr} R \circ S = \sum_{i,j} g(R \circ S(e_i \wedge e_{\bar{j}}), e_{\bar{i}} \wedge e_j),$$

where $\{e_1, \ldots, e_n, e_{\bar{1}}, \ldots, e_{\bar{n}}\}$ is a complex basis of $V^{\mathbf{C}}$ such that $g(e_i, e_{\bar{j}}) = \delta_{ij}$.

LEMMA 1. *The form* $g\colon \mathcal{R} \times \mathcal{R} \longrightarrow \mathbf{C}$ *is nondegenerate.*

PROOF. If $R \in \mathcal{R}$,

$$\begin{aligned} g(R, R) &= \sum_{i,j} g(R(e_i \wedge e_{\bar{j}}), R(e_{\bar{i}} \wedge e_j)) \\ &= \sum_{i,j} \sum_{\alpha,\beta} \sum_{\gamma,\delta} g(R(e_i \wedge e_{\bar{j}}), e_{\bar{\alpha}} \wedge e_\beta) g(R(e_{\bar{i}} \wedge e_j), e_{\bar{\gamma}} \wedge e_\delta) \\ &\qquad\qquad \cdot g(e_\alpha \wedge e_{\bar{\beta}}, e_\gamma \wedge e_{\bar{\delta}}) \\ &= \sum_{i,j} \sum_{\alpha,\beta} |g(R(e_i \wedge e_{\bar{j}}), e_{\bar{\alpha}} \wedge e_\beta)|^2 = 0 \\ &\Longleftrightarrow g(R(e_i \wedge e_{\bar{j}}), e_{\bar{\alpha}} \wedge e_\beta) = 0 \quad \text{for all } i, j, \alpha, \beta \\ &\Longleftrightarrow R(e_i \wedge e_{\bar{j}}) = \sum_{\alpha,\beta} g(R(e_i \wedge e_{\bar{j}}), e_{\bar{\alpha}} \wedge e_\beta) e_\alpha \wedge e_{\bar{\beta}} = 0 \quad \text{for all } i, j \\ &\Longleftrightarrow R \equiv 0. \end{aligned}$$

We can consider the subspace $\mathcal{B}$ of $\mathcal{R}$ consisting of those $R \in \mathcal{R}$ which satisfy: $R \in \mathcal{B}$ if and only if

$$g(R(u \wedge v), w \wedge x) - g(R(w \wedge v), u \wedge x) = 0,$$

for all $u, w \in V^{1,0}$, $v, x \in V^{0,1}$. Set $\mathcal{S} = \mathcal{B}^*$ the complement of $\mathcal{B}$ in $\mathcal{R}$ (i.e.: $g(R, S) = 0$ for all $R \in \mathcal{B}$, $S \in \mathcal{S}$).

Given $R \in \mathcal{R}$, its holomorphic sectional curvature is the real function given by

$$\sigma_R(\alpha) = -g(R\alpha, \alpha), \qquad \alpha = u \wedge \bar{u}.$$

We construct, for each $\xi \in \Lambda^{2,2}$, an operator $S_\xi \in \mathcal{S}$ as follows. Given ξ, define $S_\xi\colon \Lambda^{1,1} \longrightarrow \Lambda^{1,1}$ by

$$g(S_\xi \alpha, \beta) = g(\alpha \wedge \beta, \xi), \quad \alpha, \beta \in \Lambda^{1,1}.$$

Obviously, $S_\xi \in \mathcal{R}$. To see that $S_\xi \in \mathcal{S}$ we need the following

LEMMA 2. *Let* $\{e_1, \ldots, e_n, e_{\bar{1}}, \ldots, e_{\bar{n}}\}$ *be a basis for* $V^{\mathbf{C}}$ *such that* $g(e_i, e_{\bar{j}}) = \delta_{ij}$. *For* $1 \leqslant i < k \leqslant n$, $1 \leqslant j < l \leqslant n$, *set* $S_{i\bar{j}k\bar{l}} = S_{e_i \wedge e_{\bar{j}} \wedge e_k \wedge e_{\bar{l}}}$. *Then for* $R \in \mathcal{R}$,

$$g(R, S_{i\bar{j}k\bar{l}}) = 2\{g(R(e_{\bar{j}} \wedge e_i), e_{\bar{l}} \wedge e_k) - g(R(e_{\bar{j}} \wedge e_k), e_{\bar{l}} \wedge e_i)\}.$$

PROOF.

$$g(R, S_{i\bar{j}k\bar{l}}) = \operatorname{Tr} R \circ S_{i\bar{j}k\bar{l}} = \sum_{\alpha,\beta} g(R \circ S_{i\bar{j}k\bar{l}}(e_\alpha \wedge e_{\bar{\beta}}), e_{\bar{\alpha}} \wedge e_\beta)$$

$$= \sum_{\alpha,\beta} \sum_{\gamma,\delta} g(R(e_{\bar{\alpha}} \wedge e_\beta), e_{\bar{\gamma}} \wedge e_\delta) g(S_{i\bar{j}k\bar{l}}(e_\alpha \wedge e_{\bar{\beta}}), e_\gamma \wedge e_{\bar{\delta}})$$

$$= \sum_{\alpha,\beta} \sum_{\gamma,\delta} g(R(e_{\bar{\alpha}} \wedge e_\beta), e_{\bar{\gamma}} \wedge e_\delta) g(e_\alpha \wedge e_{\bar{\beta}} \wedge e_\gamma \wedge e_{\bar{\delta}}, e_i \wedge e_{\bar{j}} \wedge e_k \wedge e_{\bar{l}})$$

$$= 2\{g(R(e_{\bar{j}} \wedge e_i), e_{\bar{l}} \wedge e_k) - g(R(e_{\bar{j}} \wedge e_k), e_{\bar{l}} \wedge e_i)\}.$$

PROPOSITION 1. *$\xi \longrightarrow S_\xi$ maps $\Lambda^{2,2}$ isomorphically onto $\mathcal{S}$. Moreover $g(\xi, \bar{\xi}) = \frac{1}{4} g(S_\xi, S_{\bar{\xi}})$.*

PROOF. Clearly $\xi \longrightarrow S_\xi$ is a linear map from $\Lambda^{2,2}$ into $\mathcal{R}$. Since $\{e_i \wedge e_{\bar{j}} \wedge e_k \wedge e_{\bar{l}} / 1 \leqslant i < k \leqslant n,\ 1 \leqslant j < l \leqslant n\}$ is a basis for $\Lambda^{2,2}$, and the images $S_{i\bar{j}k\bar{l}}$ of the basis vectors are all in $\mathcal{S}$ ($g(R, S_{i\bar{j}k\bar{l}}) = 0$ for all $R \in \mathcal{B}$) it follows that $\xi \longrightarrow S_\xi$ maps $\Lambda^{2,2}$ into $\mathcal{S}$. In fact, Lemma 2 implies that, given $R \in \mathcal{R}$, $R \in \mathcal{B}$ if and only if $g(R, S_{i\bar{j}k\bar{l}}) = 0$ for all i, j, k, l; i.e. the $S_{i\bar{j}k\bar{l}}$ span $\mathcal{S}$ and $\xi \longrightarrow S_\xi$ maps onto $\mathcal{S}$.

If we take in the Lemma 2 $R = S_{\alpha,\bar{\beta}\gamma\bar{\delta}}$, we have

$$g(S_{\alpha\bar{\beta}\gamma\bar{\delta}}, S_{i\bar{j}k\bar{l}}) = 4\delta_{i\beta}\delta_{j\alpha}\delta_{k\delta}\delta_{l\gamma},$$

for all $\alpha < \gamma$, $\beta < \delta$, $i < k$ and $j < l$.

We suppose that $S_{\alpha\bar{\beta}\gamma\bar{\delta}} = S_{i\bar{j}k\bar{l}}$ with $1 \leqslant \alpha < \gamma \leqslant n$, $1 \leqslant \beta < \delta \leqslant n$ and $1 \leqslant i < k \leqslant n$, $1 \leqslant j < l \leqslant n$. But $g(S_{\alpha\bar{\beta}\gamma\bar{\delta}}, S_{\beta\bar{\alpha}\delta\bar{\gamma}}) = 4 = g(S_{i\bar{j}k\bar{l}}, S_{\beta\bar{\alpha}\delta\bar{\gamma}})$, thus $\delta_{i\alpha}\delta_{j\beta}\delta_{k\gamma}\delta_{l\delta} = 1$ and $\xi \longrightarrow S_\xi$ is injective.

The fact that $g(\xi, \bar{\xi}) = \frac{1}{4} g(S_\xi, S_{\bar{\xi}})$ follows from

$$g(e_i \wedge e_{\bar{j}} \wedge e_k \wedge e_{\bar{l}}, e_j \wedge e_{\bar{i}} \wedge e_l \wedge e_{\bar{k}}) = 1 = \tfrac{1}{4} g(S_{i\bar{j}k\bar{l}}, S_{j\bar{i}l\bar{k}}).$$

REMARK 1. Using the natural isomorphism between $\Lambda^{2,2}$ and its dual, the space of alternating (2,2)-forms on $V^{\mathbf{C}}$, given by the form g we can identify $\mathcal{S}$ with the space of (2,2)-forms.

PROPOSITION 2. *Let $\alpha \in \Lambda^{1,1}$. Then α is decomposable if and only if $g(S\alpha, \alpha) = 0$ for all $S \in \mathcal{S}$.*

PROOF. The necessity of the condition is clear since each $S \in \mathcal{S}$ is of the form S_ξ for some $\xi \in \Lambda^{2,2}$ and $g(S_\xi\alpha, \alpha) = g(\alpha \wedge \alpha, \xi) = 0$ for α decomposable. Conversely, let $\{e_1, \ldots, e_n, e_{\bar{1}}, \ldots, e_{\bar{n}}\}$ be a basis of $V^{\mathbf{C}}$ such that $g(e_i, e_{\bar{j}}) = \delta_{ij}$ and let $\alpha \in \Lambda^{1,1}$, $\alpha = \Sigma_{i,j}\, a_{i\bar{j}} e_i \wedge e_{\bar{j}}$, then for $1 \leqslant i < k \leqslant n$, $1 \leqslant j < l \leqslant n$, we have:

$$g(S_{i\bar{j}k\bar{l}}\alpha, \alpha) = 2(a_{j\bar{i}}a_{l\bar{k}} - a_{l\bar{i}}a_{j\bar{k}}) = 0,$$

but $a_{j\bar{i}}a_{l\bar{k}} - a_{l\bar{i}}a_{j\bar{k}} = 0$ if and only if α is decomposable.

REMARK 2. It is clear from Proposition 2, that each curvature tensor $S \in \mathcal{S}$ has holomorphic sectional curvature σ_S identically zero. Conversely, it is easily checked that this property characterizes $\mathcal{S}$.

For a subset Z of G, let

$$\mathcal{A}(Z) = \{S \in \mathcal{S} / S(P) = 0 \text{ for all } P \in Z\}.$$

Thus $\mathcal{A}(Z)$ is the subspace of $\mathcal{S}$ consisting of all elements of $\mathcal{S}$ which annihilate Z. For a finite subset $Z = \{P_1, \ldots, P_k\}$ of G, we shall denote $\mathcal{A}(\{P_1, \ldots, P_k\})$ simply by $\mathcal{A}(P_1, \ldots, P_k)$. By $\mathcal{A}(Z)^*$ with $Z \subset G$ we shall mean the complement of $\mathcal{A}(Z)$ in $\mathcal{S}$, (i.e. $g(S, S') = 0$ for all $S \in \mathcal{A}(Z)$, $S' \in \mathcal{A}(Z)^*$).

PROPOSITION 3 [4]. *Let $R \in \mathcal{R}$ and $Z \subset G$, and suppose there exists $S \in \mathcal{S}$ such that $Z \subset \operatorname{Ker}(R - S)$. Then there exists a unique $S_0 \in \mathcal{A}(Z)^*$ such that $Z \subset \operatorname{Ker}(R - S_0)$. Moreover, given any $S \in \mathcal{S}$, $Z \subset \operatorname{Ker}(R - S)$ if and only if the projection of S onto $\mathcal{A}(Z)^*$ is S_0.*

REMARK 3. Note that if $R \in \mathcal{R}$, $S \in \mathcal{S}$ and $P \in G \cap \operatorname{Ker}(R - S)$ then $\sigma_R(P) = -g(RP, P) = -g(SP, P) = \sigma_S(P) = 0$.

In particular, setting $Z(R) = \{P \in G / \sigma_R(P) = 0\}$ we see that if, for some $S \in \mathcal{S}$, the subspace $\operatorname{Ker}(R - S)$ has nonnull intersection with G, then the set $Z(R)$ of zeroes of σ_R is at least big enough to contain this intersection.

PROPOSITION 4 [4]. *Let $R \in \mathcal{R}$, and suppose there exists $S \in \mathcal{S}$ such that $Z(R) = G \cap \operatorname{Ker}(R - S)$. Then there exists a unique $S_0 \in \mathcal{A}(Z(R))^*$ such that $Z(R) = G \cap \operatorname{Ker}(R - S_0)$.*

2. A basis for the normal space to G in $\Lambda^{1,1}$.

LEMMA 3. *Let $P \in G$, and let $\{e_1, \ldots, e_n, e_{\bar{1}}, \ldots, e_{\bar{n}}\}$ be a basis for $V^{\mathbb{C}}$ such that $g(e_i, e_{\bar{j}}) = \delta_{ij}$. Then*

$$\{P\} \cup \{S_{i\bar{j}k\bar{l}}(P) / 1 \leqslant i < k \leqslant n,\ 1 \leqslant j < l \leqslant n\}$$

spans the normal space to $G \subset \Lambda^{1,1}$ at P. If the basis is chosen so that $P = e_1 \wedge e_{\bar{1}}$ then $\{P\} \cup \{S_{1\bar{1}k\bar{l}}(P) / 1 \leqslant k, l \leqslant n\}$ is a basis for this normal space which verifies

$$g(P, \bar{P}) = 1, \qquad g(P, S_{1\bar{1}k\bar{l}}(P)) = 0$$

and

$$g(S_{1\bar{1}k\bar{l}}(P), S_{\bar{1}1\bar{p}q}(P)) = \begin{cases} 1 & \text{if } (k, l) = (p, q), \\ 0 & \text{if } (k, l) \neq (p, q). \end{cases}$$

PROOF. By Proposition 2,

$$G = \{\alpha \in \Lambda^{1,1}/g(\alpha, \bar{\alpha}) = 1, g(S_{i\bar{j}k\bar{l}}(\alpha), \bar{\alpha}) = 0,$$
$$\text{and } \alpha + \bar{\alpha} = 0 \text{ for all } i < k, j < l\}.$$

Since the functions $\alpha \longrightarrow g(\alpha, \bar{\alpha})$ and $\alpha \longrightarrow g(S_{i\bar{j}k\bar{l}}(\alpha), \bar{\alpha})$ are constant on G, their gradients $2P$ and $2S_{i\bar{j}k\bar{l}}(P)$ at $P \in G$ must be normal to G at P. To see that they span the normal space N_p of G at P, consider first the case where $P = e_1 \wedge e_{\bar{1}}$. Then, for $i < k, j < l$,

$$S_{i\bar{j}k\bar{l}}(P) = \begin{cases} -e_k \wedge e_{\bar{l}} & \text{for } i, j = 1, \\ 0 & \text{for } i, j \neq 1. \end{cases}$$

Moreover,

$$g(P, \bar{P}) = 1 \quad \text{and} \quad g(S_{1\bar{1}k\bar{l}}(P), S_{\bar{1}1\bar{p}q}(P)) = \begin{cases} 1 & \text{if } (k, l) = (p, q), \\ 0 & \text{if } (k, l) \neq (p, q). \end{cases}$$

In this case, $\{P\} \cup \{S_{1\bar{1}k\bar{l}}(P)/1 < k, l \leqslant n\}$ is a linearly independent subset in N_p. The number $1 + (n-1)^2$ of elements in this set is equal to the complex codimension $n^2 - 2(n-1)$ of G in $\Lambda^{1,1}$ which in turn is equal to the dimension of N_p. Hence $\{P\} \cup \{S_{1\bar{1}k\bar{l}}(P)/1 < k, l \leqslant n\}$ is a basis for N_p.

In the general case, let $\{e_1, \ldots, e_n, e_{\bar{1}}, \ldots, e_{\bar{n}}\}$ be an arbitrary basis for $V^{\mathbf{C}}$ such that $g(e_i, e_{\bar{j}}) = \delta_{ij}$ and let $\{e'_1, \ldots, e'_n, e'_{\bar{1}}, \ldots, e'_{\bar{n}}\}$ be one such that $P = e'_1 \wedge e'_{\bar{1}}$. Let $\{S'_{i\bar{j}k\bar{l}}/1 \leqslant i < k \leqslant n, 1 \leqslant j < l \leqslant n\}$ and $\{S'_{i\bar{j}k\bar{l}}\}$ be the corresponding bases for $\mathcal{S}$. Then, from above, we know that $\{P\} \cup \{S'_{1\bar{1}k\bar{l}}(P)/1 < k, l \leqslant n\}$ spans N_p. But each $S'_{1\bar{1}k\bar{l}}$ is a linear combination of the $S_{i\bar{j}k\bar{l}}$ and, hence each $S'_{1\bar{1}k\bar{l}}(P)$ is a linear combination of the $S_{i\bar{j}k\bar{l}}(P)$. Thus $\{P\} \cup \{S_{i\bar{j}k\bar{l}}(P)/1 \leqslant i < k \leqslant n, 1 \leqslant j < l \leqslant n\}$ spans N_p.

PROPOSITION 5. *Let $R \in \mathcal{R}$ and suppose $P \in G$ is a critical zero of σ_R. Then there exists $S \in \mathcal{S}$ such that $P \in \operatorname{Ker}(R - S)$.*

PROOF. Let $\{e_1, \ldots, e_n, e_{\bar{1}}, \ldots, e_{\bar{n}}\}$ be a basis for $V^{\mathbf{C}}$ such that $g(e_i, e_{\bar{j}}) = \delta_{ij}$ and $P = e_1 \wedge e_{\bar{1}}$. Since P is a critical point of σ_R, and σ_R is the restriction to G of the function $\alpha \mapsto g(R(\alpha), \bar{\alpha})$, the gradient $2R(P)$ of this function at P must be normal to G at P. By Lemma 3, this implies that

$$RP = \lambda P + \sum_{1<k,l} \mu_{k\bar{l}} S_{1\bar{1}k\bar{l}}(P),$$

for some $\lambda, \mu_{k\bar{l}} \in \mathbf{C}$. But $\lambda = -g(RP, P) = \sigma_R(P) = 0$, so $P \in \operatorname{Ker}(R - S)$ where $S = \Sigma_{1<k,l}\, \mu_{k\bar{l}} S_{1\bar{1}k\bar{l}}$.

COROLLARY 1. *Let $R \in \mathcal{R}$ and suppose $P \in G$ is a critical zero of σ_R. Then there exists a unique $S \in \mathcal{A}(Z)^*$ such that $P \in \operatorname{Ker}(R - S)$.*

REMARK 4. The operator S constructed in the proof of Proposition 5 is in fact the unique $S \in \mathcal{A}(P)^*$ such that $P \in \operatorname{Ker}(R - S)$. Indeed, by Lemma 2, together with the fact that each $S' \in \mathcal{S}$ is an S_ω for some alternating (2, 2)-form ω on V, we have

$$g(S', S_{1\bar{1}k\bar{l}}) = 4g(S'(e_1 \wedge e_{\bar{1}}), e_k \wedge e_{\bar{l}}),$$

and this is zero for all $S' \in \mathcal{A}(P)$; thus $S_{1\bar{1}k\bar{l}} \in \mathcal{A}(P)^*$ for all $1 < k, l \leqslant n$.

Note also that, since $\{S_{1\bar{1}k\bar{l}}/1 < k, l \leqslant n\}$ is linearly independent, the numbers $\mu_{k\bar{l}}$ above are uniquely determined by the components of R relative to the basis $(e_i, e_{\bar{j}})$:

$$\begin{aligned}
\mu_{k\bar{l}} &= g\Big(\sum_{1<p,q} \mu_{p\bar{q}} e_p \wedge e_{\bar{q}}, e_{\bar{k}} \wedge e_l\Big) \\
&= -g\Big(\sum_{1<p,q} \mu_{p\bar{q}} S_{1\bar{1}p\bar{q}}(e_1 \wedge e_{\bar{1}}), e_{\bar{k}} \wedge e_l\Big) \\
&= -g(R(e_1 \wedge e_{\bar{1}}), e_{\bar{k}} \wedge e_l).
\end{aligned}$$

3. The main results.

PROPOSITION 6. *Let* $\dim_{\mathbf{C}} V^{\mathbf{C}} = 4$, *and suppose $R \in \mathcal{R}$ is such that $\sigma_R \geqslant 0$ and $Z(R) \neq 0$. Then there exists a unique $S \in \mathcal{S}$ such that $Z(R) = G \cap \operatorname{Ker}(R - S)$.*

For the proof we proceed in the same way as the real case, where $\{*\} = \{S_{1\bar{1}2\bar{2}}\}$, $\mu, \mu_i \in \mathbf{C}$ and $l = (g(P_1 + P_2, \overline{P_1 + P_2}))^{1/2}$.

COROLLARY 2. *Let* $\dim_{\mathbf{C}} V^{\mathbf{C}} = 4$ *and $R \in \mathcal{R}$, and λ denote the minimum (or maximum) value of σ_R. Then there exists a unique $S \in \mathcal{S}$ such that*

$$\{P \in G/\sigma_R(P) = \lambda\} = G \cap \operatorname{Ker}(R - \lambda I - S),$$

where I is defined by: $I(P) = P$ for all $P \in \Lambda^{1,1}$.

PROOF. Follows immediately from Proposition 6 replacing R in the proposition by $R - \lambda I$ (or, in the case where λ is the maximum of σ_R, by $\lambda I - R$).

LEMMA 4. *Let $R \in \mathcal{R}$ be such that $\sigma_R \geqslant 0$, and suppose $P, Q \in Z(R)$. Then there exists $S \in \mathcal{S}$ such that $\{P, Q\} \subset \operatorname{Ker}(R - S)$.*

PROOF. Choose a basis $\{e_1, \ldots, e_n, e_{\bar{1}}, \ldots, e_{\bar{n}}\}$ for $V^{\mathbf{C}}$ such that

$g(e_i, e_{\bar{j}}) = \delta_{ij}$ and $P = e_1 \wedge e_{\bar{1}}$ and Q is contained in the span of $\{e_1, e_2, e_{\bar{1}}, e_{\bar{2}}\}$, so that $Q = \Sigma_{i,j\leqslant 2}\, q_{i\bar{j}}\, e_i \wedge e_{\bar{j}}$ for some $q_{i\bar{j}} \in \mathbf{C}$. Since Q is a critical point (a minimum) of σ_R, $RQ = \frac{1}{2}(\text{grad } \sigma_R)(Q)$ is normal to G at Q, so by Lemma 3,

$$RQ = \sum_{i<k,j<l} \nu_{i\bar{j}k\bar{l}} S_{i\bar{j}k\bar{l}}(Q), \tag{1}$$

for some $\nu_{i\bar{j}k\bar{l}} \in \mathbf{C}$ (the component of RQ in the direction of Q is zero since $g(RQ, Q) = -\sigma_R(Q) = 0$). Note that the $\nu_{i\bar{j}k\bar{l}}$ are not uniquely determined, since the $S_{i\bar{j}k\bar{l}}(Q)$ are not linearly independent.

Similarly,

$$RP = \sum_{1<k,l} \mu_{1\bar{1}k\bar{l}} S_{1\bar{1}k\bar{l}}(P), \tag{2}$$

where now the $\mu_{1\bar{1}k\bar{l}}$ are uniquely determined, since the $S_{1\bar{1}k\bar{l}}$ verifies

$$g(S_{1\bar{1}k\bar{l}}(P), S_{\bar{1}1\bar{p}q}(P)) = \delta_{kp}\delta_{lq}.$$

Moreover, by Remark 4, $S_1 = \Sigma\, \mu_{1\bar{1}k\bar{l}} S_{1\bar{1}k\bar{l}}$ is the unique operator in $\mathcal{A}(P)^*$ such that $P \in \operatorname{Ker}(R - S_1)$. Thus, by Proposition 3, it suffices to construct an $S_2 \in \mathcal{S}$ such that $Q \in \operatorname{Ker}(R - S_2)$ and such that the projection of S_2 onto $\mathcal{A}(P)^*$ is S_1. But

$$g(S_{i\bar{j}k\bar{l}}, S_{\bar{\alpha}\beta\bar{\gamma}\delta}) = \begin{cases} 4 & \text{if } (\alpha, \beta, \gamma, \delta) = (i, j, k, l) \text{ for } i < k,\ j < l, \\ & \qquad \alpha < \gamma \text{ and } \beta < \delta, \\ 0 & \text{if } (\alpha, \beta, \gamma, \delta) \neq (i, j, k, l). \end{cases}$$

$S_{1\bar{1}k\bar{l}} \in \mathcal{A}(P)^*$ for $1 < k, l \leqslant n$ and $S_{i\bar{j}k\bar{l}} \in \mathcal{A}(P)$ for $i, j \neq 1$. So the projection into $\mathcal{A}(P)^*$ of $\Sigma_{i<k,j<l}\nu_{i\bar{j}k\bar{l}}S_{i\bar{j}k\bar{l}}$ is just $\Sigma_{1<k,l}\nu_{1\bar{1}k\bar{l}}S_{1\bar{1}k\bar{l}}$. Thus we must show that we can choose $\nu_{i\bar{j}k\bar{l}} \in \mathbf{C}$ such that

$$RQ = \sum_{i<k,j<l} \tilde{\nu}_{i\bar{j}k\bar{l}} S_{i\bar{j}k\bar{l}} \quad \text{and} \quad \tilde{\nu}_{1\bar{1}k\bar{l}} = \mu_{1\bar{1}k\bar{l}} \quad \text{for } 1 < k, l.$$

The remainder of the proof is parallel to that of Lemma 5.1 of [4] using the fact that the $\nu_{i\bar{j}k\bar{l}}$ are not uniquely determined.

LEMMA 5 [4]. *Let $Z \subset G$. Then there exists a finite subset $\{P_1, \ldots, P_k\}$ of Z such that if $R \in \mathcal{R}$ and $P_i \in$* Ker (R) *for all $i \leqslant k$, then $Z \subset$* Ker(R).

LEMMA 6. *Let X be a complex vector space with a nondegenerate complex symmetric bilinear form, and $X_i (1 \leqslant i \leqslant k)$ subspaces of X such that $X = \Sigma_{i=1}^k X_i$. Let Π_i: $X \longrightarrow X_i$ and Π_{ij}: $X \longrightarrow X_i \cap X_j$ $(1 \leqslant i, j \leqslant k)$ denote orthogonal projections, and $x_i \in X_i (1 \leqslant i \leqslant k)$ be such that $\Pi_{ij}x_i = \Pi_{ij}x_j$ for all $i \neq j$. Then there exists a unique $x \in X$ such that $\Pi_i x = x_i$ for all i.*

PROOF. By induction on k.

PROPOSITION 7. *Let $R \in \mathcal{R}$ be such that $\sigma_R \geqslant 0$. Then there exists $S \in \mathcal{S}$ such that $Z(R) = G \cap \operatorname{Ker}(R - S)$.*

The proof is similar to that of Theorem 5.4 of [4].

COROLLARY 3. *Let $R \in \mathcal{R}$ and let λ denote the minimum (or maximum) value of σ_R. Then there exists $S \in \mathcal{S}$ such that*

$$\{P \in G / \sigma_R(P) = \lambda\} = G \cap \operatorname{Ker}(R - \lambda I - S).$$

PROOF. Immediate from Proposition 7 upon replacing R by $R - \lambda I$ (or, in the maximum case, by $\lambda I - R$).

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